"Properties of Linear Phase (LP) FIR filters" motivates the four types of LP FIR filters. Linear phase FIR filters have constant group delay which means that all passed frequencies experience the same delay, thus avoiding phase dispersion, e.g., 'smeared' attack transients in percussive sounds. When the cases of symmetry and anti-symmetry are combined with even and odd filter order M, we get four types of linear phase FIR filters. Examples are given for each type, and space is provided for students to graph the resulting frequency responses. In so doing, they see that frequency responses for each of these types have particular characteristics and shapes. Causal FIR filters are always stable because their poles are located at z = 0 in the complex plane. Therefore, it's the location of their zeros that differentiate the 4 types. Constraints on the zeros imposed by each type limit the kinds of frequency responses that can be achieved, a significant finding for designing LP FIR filters. Handwritten derivations of the Type 1 and Type 2 frequency response formulas, as well as derivations of Eqn. 1 and Eqn. 2, accompanied the handout.

Properties of Linear Phase (LP) FIR filters

NOTE: L = length of the FIR filter impulse response h[n], and M = L-1 = order of the FIR filter.

The frequency response of the FIR filter is given by $H(\omega) = \sum_{0}^{M} h[n]e^{-j\omega n}, \quad -\pi \le \omega \le \pi$. Recall the system function $H(z) = \sum_{0}^{M} h[n]z^{-n} = z^{-M}\sum_{0}^{M} h[n]z^{M-n}$ → M poles at the origin and M zeros located anywhere in the z-plane.

We will see that linear phase imposes constraints on the locations of the M zeros.

We have two possible cases for linear phase (LP) filters:

<u>Case A</u>: h[n] symmetric so that $\measuredangle H(\omega) = -\alpha\omega, \quad -\pi \le \omega \le \pi$, where α = constant group delay=$\frac{M}{2}$, the index (or center) of symmetry for h[n]. The constant α also gives the group delay of H(ω). Why?

<u>Case B</u>: h[n] anti-symmetric so that $\measuredangle H(\omega) = \beta - \alpha\omega, \quad -\pi \le \omega \le \pi$. Here, the group delay is also constant since $\frac{d\measuredangle H(\omega)}{d\omega} = -\alpha$, a constant group delay. As in Case A, α=$\frac{M}{2}$ is also known as the index (or center) of symmetry for h[n]. $\beta = \pm\frac{\pi}{2}$.

The group delay offers another graphical representation conveying useful information about the frequency response of a filter, specifically about the phase of H(ω).

The group delay $\tau(\omega) = -\frac{d\measuredangle H(\omega)}{d\omega}$ gives the delay, in samples, introduced by the system to a sinusoid of frequency ω.

The MATLAB command: [Gd,w] = GRPDELAY(b,a,N) returns length N vectors Gd and w containing the group delay and the frequencies (in radians) at which it is evaluated. The frequency response is evaluated at N points equally spaced around the upper half of the unit circle. If you don't specify N, it defaults to 512.

<u>Causal FIR filters with Linear Phase</u>

If h[n] is causal and symmetric, i.e., if $h[n]=\begin{cases} h[M-n], & 0 \le n \le M \\ 0, & \text{else} \end{cases}$, then

$H(\omega)=A_e(\omega)e^{-j\omega M/2}$, where $A_e(\omega)$ is a real, even, periodic function of ω.   Eqn. (*)

Similarly, if h[n] is causal and anti-symmetric, i.e., if $h[n]=\begin{cases} -h[M-n], & 0 \le n \le M \\ 0, & \text{else} \end{cases}$, then

$H(\omega)=A_o(\omega)e^{-j\omega M/2+j\pi/2}$, where $A_o(\omega)$ is a real, odd, periodic function of ω.   Eqn. (**)

L = length of h[n] = M+1 samples in both cases.

When the cases of symmetry and anti-symmetry are combined with even and odd M, we get four types of linear phase FIR filters. Frequency responses for each of these types have particular characteristics and shapes!

Type I FIR Linear Phase Systems (causal h[n] symmetric, M even)

Symmetry condition: h[n] = h[M-n],   0 ≤ n ≤ M with M even integer. Time delay M/2 is also an integer.

FIR Frequency response: $H(\omega)=\sum_{0}^{M} h[n]e^{-j\omega n}, \quad -\pi \le \omega \le \pi$ Eqn. (***)

Apply symmetry condition to Eqn. (***) to get the Type I LP FIR frequency response:

$H(\omega)=e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right)$, where

a[0] = h[M/2],

a[k] = 2h[(M/2)-k],   k = 1,2, … , M/2. Thus we see that H(ω) has the form of Eqn. (*) given above, where

$$A_e(\omega)=\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right).$$

EX 1: h[n] = {<u>**1**</u>,1,1,1,1}. What is the center of symmetry?

EX 2: h[n] = {<u>**1**</u>, 1, 1}.   Find H(ω). Graph |H(ω)| and ∠ H(ω) .

Type II FIR Linear Phase Systems (causal h[n] symmetric, M odd)

Symmetry condition: $h[n] = h[M-n]$, $0 \le n \le M$, with M odd integer. Time delay M/2 is an integer plus one-half.

Apply symmetry condition to Eqn. (***) to get the Type II LP FIR frequency response:

$$H(\omega) = e^{-j\omega M/2}\left(\sum_{k=1}^{(M+1)/2} b[k]\cos[\omega(k-\frac{1}{2})]\right)$$, where

$b[k] = 2h[(M+1)/2 - k]$, $k = 1,2, \dots, (M+1)/2$. Again, we see that $H(\omega)$ has the form of Eqn. (*) given above.

EX 3: h[n] = {**1**,1,1,1,1,1}. Find the center of symmetry.

EX 4: h[n] = {**1**, 1}. Find $H(\omega)$. Graph $|H(\omega)|$ and $\angle H(\omega)$.

Type III FIR Linear Phase Systems (causal anti-symmetric h[n], M even)

Anti-symmetry condition: $h[n] = -h[M-n]$, $0 \le n \le M$ with M even integer. Time delay M/2 is an integer.

Apply anti-symmetry condition to Eqn. (***) to get the Type III LP FIR frequency response:

$$H(\omega) = je^{-j\omega M/2}\left(\sum_{k=0}^{M/2} c[k]\sin\omega k\right)$$, where

$c[k] = 2h[(M/2)-k]$, $k = 1,2, \dots, M/2$. Thus we see that $H(\omega)$ has the form of Eqn. (**).

EX 5: h[n] = {**1**,0,-1}. Center of symmetry = ___________. Find $H(\omega)$. Graph $|H(\omega)|$ and $\angle H(\omega)$.

Type IV FIR Linear Phase Systems (causal anti-symmetric h[n], M odd)

Anti-symmetry condition: $h[n] = -h[M-n]$, $0 \le n \le M$ with M odd integer. Time delay M/2 is an integer plus one-half.

Apply anti-symmetry condition to Eqn. (***) to get the Type IV LP FIR frequency response:

$$H(\omega) = je^{-j\omega M/2}\left(\sum_{k=1}^{(M+1)/2} d[k]\sin\omega(k-\frac{1}{2})\right)$$, where

d[k] = 2h[(M+1)/2 - k], k = 1,2, … , (M+1)/2. Again, we see that H(ω) has the form of Eqn. (**) given above.

EX 6: {**<u>1</u>**,-1}. Determine the center of symmetry: _______. Find H(ω). Graph |H(ω)| and ∠ H(ω) .

Locations of Zeros for FIR LP Systems

<u>Types I and II</u>

System function H(z) for causal FIR filters: $H(z) = \sum_{0}^{M} h[n]z^{-n}$. For symmetric cases (types I and II), use

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M \\ 0, & \text{else} \end{cases}$$ to express H(z) as

$$H(z)_{symmetric} = \sum_{n=0}^{M} h[M-n]z^{-n} = z^{-M}H(z^{-1})\,.$$ **Eqn. (1)** Symmetric h[n]: for FIR types I and II

Deduce from Eqn. (1):

a. If z_0 is a zero of H(z), then $H(z_0) = z_0^{-M}H(z_0^{-1}) = 0$. This implies that:

i. If $z_0 = re^{j\theta}$ is a zero of H(z), then $z_0^{-1} = r^{-1}e^{-j\theta}$ is also a zero of H(z).

b. When h[n] is real and z_0 is a zero of H(z), then $z_0^* = re^{-j\theta}$ will also be a zero of H(z), and due to part a(i), $(z_0^*)^{-1} = r^{-1}e^{j\theta}$ will also be a zero of H(z).

c. Therefore, when h[n] real, each complex zero NOT on the unit circle will be part of a set of four conjugate reciprocal zeros of the form $(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})(1-r^{-1}e^{j\theta}z^{-1})(1-r^{-1}e^{-j\theta}z^{-1})$.

d. If a zero of H(z) is on the unit circle, i.e., $z_0 = e^{j\theta}$, then $z_0^{-1} = e^{-j\theta} = z_0^*$. Therefore, zeros on the unit circle come in pairs of the form $(1-e^{j\theta}z^{-1})(1-e^{-j\theta}z^{-1})$.

e. If a zero of H(z) is real and not on the unit circle, the reciprocal will also be a zero of H(z), and H(z) will have factors of the form $(1-rz^{-1})(1-r^{-1}z^{-1})$ or factors of the form $(1+rz^{-1})(1+r^{-1}z^{-1})$.

f. The case of a zero at z = -1 needs to be examined, especially for Type II LP FIR filters. From Eqn. (1), H(-1) = $(-1)^M$ H(-1). If M is even (Type I), then get an identity. But if M is odd, H(-1) = -H(-1): the only solution is H(-1) must be zero. Therefore, for symmetric h[n] with M odd, the system function H(z) MUST have a zero at z = -1.

<u>Types III and IV</u>

If h[n] anti-symmetric (types III and IV), then can show that

$$H(z)_{anti-symmetric} = -\sum_{n=0}^{M} h[M-n]z^{-n} = -z^{-M}H(z^{-1}).$$ **Eqn. (2)** Anti-symmetric h[n]: FIR types III and IV.

Deduce from Eqn. (2):

a. The zeros of H(z) for the anti-symmetric case are constrained in the same way as the zeros for the symmetric case.

b. In the anti-symmetric case, however, both z = 1 and z = -1 hold special interest.

c. If z = 1, Eqn. (2) becomes H(1) = -H(1) → H(z) MUST have a zero at z = 1 for both M even and M odd.

d. If z = -1, Eqn. (2) gives H(-1) = $(-1)^{-M+1}$ H(-1). If M is even, then H(-1) = -H(-1). Therefore, z = -1 MUST be a zero of H(z) if M is even (so what type of linear phase FIR filter would always have a zero at z = -1?).

These constraints on the zeros impose limitations on the types of frequency responses that can be achieved with FIR LP filters. As designers, we need to take into account these constraints. For example, if you are designing a filter with an anti-symmetric h[n] *that has to allow the highest frequencies for a given Fs*, would you choose M to be even or odd?